DERIVATION OF RELATIONSHIPS FOR CALCULATING HYDRODYNAMIC CHARACTERISTICS OF LIQUID FILMS ON A ROTATING DISK

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Expressions are obtained for calculating film flow rates of liquids with complex rheological behavior on the surface of a rotating disk.

Film flow of a liquid on the surface of a rotating disk is the basis of many technological processes [1-3], so that calculation of the hydrodynamic properties of such flows is of great importance.

This question has been investigated in a number of studies, which derived relationships for Newtonian [4-5], power-law [6-8], and viscoplastic liquids [9]. However the rheological behavior of real media cannot always be described within the framework of those models [10]. In connection with this, the goal of the present study is to investigate the flow of a medium, the rheological behavior of which is described by the more general equation proposed and justified by Z. P. Shul'man [11]

$$\tau^{1/n} = \tau_0^{1/n} + (K\gamma)^{1/m} . \tag{1}$$

This equation combines plasticity and nonlinear viscosity to various degrees and generalizes the majority of known rheological models: that of Newton ($\tau_0 = 0$; m = n = 1), Shevedov—Bingman (m = n = 1), Balkley—Herschel (n = 1), power law ($\tau_0 = 0$), Casson (m = n = 2), etc. [10].

To study liquid flow over the surface of a rotating disk we will make use of the Navier—Stokes equation and the continuity equation, written in a cylindrical coordinate system moving with the disk [12]. In solving the problem we will assume the motion to be settled and neglect the effect of the Coriolis force, gravity, and surface tension as compared to the centrifugal force. The rheological properties and density of the medium will be assumed to depend on time and radius.

The numerically small ratio of the characteristic film thickness H to the characteristic radius R and the presence of axial symmetry permit significant simplification of the system of equations of motion. After transition to dimensionless quantities, elimination of derivatives with respect to polar angles, and terms of order of smallness greater than or equal to H/R ≪ 1, we obtain

$$\frac{\partial \tau^*}{\partial z^*} = -\operatorname{Re} \rho^* r^*, \tag{2}$$

$$\frac{d\rho^*}{dt^*} + \rho^* \left(\frac{\partial u^*}{\partial r^*} + \frac{u^*}{r^*} + \frac{\partial w^*}{\partial z^*} \right) = 0. \tag{3}$$

Integration of Eq. (2) yields

$$\tau^* = -\operatorname{Re} \rho^* r^* z^* + C_1(r^*). \tag{4}$$

Since the ratio of the dynamic viscosity of air to the viscosity of the liquid medium is small, air friction on the film surface can be neglected, i.e., we may take

$$\tau^* = 0 \quad \text{at} \quad z^* = h^*. \tag{5}$$

It follows from Eqs. (4), (5) that

$$\tau^* = \operatorname{Re} \rho^* r^* (h^* - z^*). \tag{6}$$

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Flow does not develop within the liquid until $\tau < \tau_0$ [9]. The coordinate z_0, above which flow is absent, can be determined from the equation

$$\tau_0^* = \mathrm{Re}\,\rho^* r^* (h^* - z_0^*), \tag{7}$$

whence

$$z_0^* = h^* - \frac{1}{\mathrm{Re}}\,\frac{\tau_0^*}{\rho^* r^*}. \tag{8}$$

Using Eqs. (1) and (6), we obtain

$$\frac{\partial u^*}{\partial z^*} = \frac{H}{R}\,\frac{1}{K^*}\left\{(\mathrm{Re}\,\rho^* r^*)^{1/n}\left[(h^* - z^*)^{1/n} - \left(\frac{1}{\mathrm{Re}}\,\frac{\tau_0^*}{\rho^* r^*}\right)^{1/n}\right]\right\}^m. \tag{9}$$

Integration of Eq. (9) for $z \le z_0$ with consideration of the condition of liquid "adhesion" to the disk surface ($u^* = 0$ at $z^* = 0$) leads to the equation

$$u^* = A\sum_{l=0}^{m} B_l\left[h^{*\frac{m-l}{n}+1} - (h^* - z^*)^{\frac{m-l}{n}+1}\right], \tag{10}$$

where

$$A = \frac{H}{R}\mathrm{Re}^{m/n}\frac{(\rho^* r^*)^{m/n}}{K^*};\quad B_l = (-1)^l\,\frac{m!}{l!\,(m-l)!}\,\frac{n}{m+n-l}\left(\frac{1}{\mathrm{Re}}\,\frac{\tau_0^*}{\rho^* r^*}\right)^{l/n}.$$

For the velocity of the upper layer of the film u_0^* at $z^* > z_0^*$

$$u_0^* = A\sum_{l=0}^{m} B_l\left[h^{*\frac{m-l}{n}+1} - \left(\frac{1}{\mathrm{Re}}\,\frac{\tau_0^*}{\rho^* r^*}\right)^{\frac{m-l}{n}+1}\right]. \tag{11}$$

In special cases for appropriate τ_0^*, m, and n Eq. (10) transforms to the expressions obtained in [4-9] for Newtonian, power-law, and viscoplastic liquids.

To find the axial velocity component we make use of the continuity equation, writing the same for $z^* \le z_0^*$ in the form

$$w^* = -\int_0^{z^*}\left(\frac{1}{\rho^*}\,\frac{d\rho^*}{dt^*} + \frac{\partial u^*}{\partial r^*} + \frac{u^*}{r^*}\right)dz^*. \tag{12}$$

For $z^* > z_0^*$, dividing the integration region into the two intervals $[0, z_0^*]$ and $[z_0^*, z^*]$ we obtain

$$w^* = -\int_0^{z_0^*}\left(\frac{1}{\rho^*}\,\frac{d\rho^*}{dt^*} + \frac{\partial u^*}{\partial r^*} + \frac{u^*}{r^*}\right)dz^* - \\ -\int_{z_0^*}^{z^*}\left(\frac{1}{\rho^*}\,\frac{d\rho^*}{dt} + \frac{\partial u_0^*}{\partial r^*} + \frac{u_0^*}{r^*}\right)dz^*. \tag{13}$$

After integration we find

$$w^* = -A\sum_{l=0}^{m}\left\{B_l\left(f_1^l\frac{\partial h^*}{\partial r^*} + f_2^l\right)\right\} - f_3, \tag{14}$$

where for $z^* \le z_0^*$

$$f_1^l = \alpha z^* h^{*\alpha-1} + (h^* - z^*)^\alpha - h^{*\alpha}; \tag{15}$$

$$f_2^l = g_1\left[z^* h^{*\alpha} + \frac{1}{\alpha+1}\left((h^* - z^*)^{\alpha+1} - h^{*\alpha+1}\right)\right]; \tag{16}$$

$$f_3 = \frac{1}{\rho^*}\int_0^{z^*}\left(\frac{\partial\rho^*}{\partial t^*} + u^*\frac{\partial\rho^*}{\partial r^*}\right)dz^*; \tag{17}$$

for $z^* > z_0^*$

$$f_1^l = \alpha z^* h^{*\alpha-1} + \lambda^\alpha - h^{*\alpha};$$

$$f_2^l = g_1\left[h^{*\alpha} z_0^* + \frac{1}{\alpha+1}(\lambda^{\alpha+1} - h^{*\alpha+1}) + (h^{*\alpha} - \lambda^\alpha)(z^* - z_0^*)\right] - \tag{18}$$

$$- g_2\alpha\lambda^\alpha(z^* - z_0^*);$$

$$f_3 = \frac{1}{\rho^*}\left\{\int_0^{z_0^*}\left(\frac{\partial\rho^*}{\partial t^*} + u^*\frac{\partial\rho^*}{\partial r^*}\right)dz^* + \right. \tag{19}$$

$$\left. + \int_{z_0^*}^{z^*}\left(\frac{\partial\rho^*}{\partial t^*} + u_0^*\frac{\partial\rho^*}{\partial r^*}\right)dz^*\right\}; \tag{20}$$

$$g_1 = \frac{m}{n}\frac{2}{r^*} + \frac{m}{n}\frac{1}{\rho^*}\frac{\partial\rho^*}{\partial r^*} - \frac{1}{K^*}\frac{\partial K^*}{\partial r^*} + \frac{l}{n}g_2;$$

$$g_2 = \frac{1}{\tau_0^*}\frac{\partial\tau_0^*}{\partial r^*} - \frac{1}{\rho^*}\frac{\partial\rho^*}{\partial r^*} - \frac{1}{r^*}; \tag{21}$$

$$\alpha = \frac{m-l}{n} + 1;\quad \lambda = \frac{1}{\mathrm{Re}}\frac{\tau_0^*}{\rho^* r^*}. \tag{22}$$

In many technological processes, in particular, in deposition of polymer coatings on a rotating substrate, calculation of the film thickness is important.

The axial component of the displacement rate of the film profile boundary can be represented in the form

$$w^*(h^*) = \frac{\partial h^*}{\partial t^*} + u_0^*\frac{\partial h^*}{\partial r^*}. \tag{23}$$

With consideration of Eq. (14) we obtain

$$\frac{\partial h^*}{\partial t^*} = -A\sum_{l=0}^{m} B_l\left\{F_1\frac{\partial h^*}{\partial r^*} + F_2\right\} - F_3, \tag{24}$$

where $F_1 = f_1^\ell|_{z^*=h^*} + h^{*\alpha} - \lambda^\alpha$; $F_2 = f_2^\ell|_{z^*=h^*}$; $F_3 = f_3|_{z^*=h^*}$, f_1^ℓ, f_2^ℓ and f_3 being defined by Eqs. (18), (19), (20) respectively.

The compositions used for formation of polymer coatings contain organic solvents, which evaporate intensely when deposited. Therefore in determining the ratio of the current polymer coating thickness to the axial velocity component, caused by flow of the composite, it is necessary to add a component related to evaporation of the solvents:

$$w_{\mathrm{ev}} = -j/\rho_{\mathrm{s}}. \tag{25}$$

The density of the diffusion flux j depends on the medium flow regime above the surface of the coating being formed.

At values of the number $\mathrm{Re}_c = \omega r^2/\nu < 10^4$ flow over the disk by the medium is laminar in character [13, 14] and the density of the diffusion flux is equal to [14]

$$j = 0.62 D c_0 \nu^{-1/2}\omega^{1/2}\mathrm{Pr}_d^{1/3}. \tag{26}$$

In the turbulent regime ($\mathrm{Re}_c > 10^4$) the density of the diffusion flux can be estimated with the expression [14]

$$j \approx 10^{-2} c_0 \nu^{-0.2}\omega^{0.8} r^{0.6}\mathrm{Pr}_d^{-3/4}. \tag{27}$$

For gases and vapors $\nu \approx D$ [15, 16]. Consequently, in Eqs. (26) and (27) we can take $\mathrm{Pr}_d \approx 1$. As is evident from these expressions, for laminar flow of the medium above the coating being formed j is independent of radius. For the turbulent regime, with increase in radial coordinate the intensity of evaporation increases and the film surface ceases to be equiaccessible in a diffusion sense.

In deposition of polymer coatings the diffusion flux density affects the solvent concentration, for determination of which we use the equation

$$\frac{d(h^*\varphi)}{dt^*} = \varphi_0 w^* + w^*_{\mathrm{ev}} . \tag{28}$$

The first term on the right side of Eq. (28) reflects the decrease in quantity of solvent related to flow of the composition, while the second is the result of diffusion into the surrounding medium.

From Eq. (28) we obtain

$$\varphi = \varphi_0 + (1 - \varphi_0)\frac{w^*_{\mathrm{ev}} t^*}{h^*} . \tag{29}$$

Considering that

$$\rho^* = \rho^*_{\mathrm{s}}\varphi + \rho^*_{\mathrm{n}}(1 - \varphi), \tag{30}$$

we find

$$\frac{\partial\rho^*}{\partial r^*} = \Delta\rho^* \frac{\partial\varphi}{\partial r^*} , \tag{31}$$

$$\frac{d\rho^*}{dt^*} = \Delta\rho^* \left(\frac{\partial\varphi}{\partial t^*} + u^* \frac{\partial\varphi}{\partial r^*}\right) , \tag{32}$$

where $\Delta\rho^* = \rho_s^* - \rho_n^*$.

Using Eq. (29) to determine φ, we obtain

$$\frac{\partial\varphi}{\partial r^*} = (1 - \varphi_0)\left[\frac{t^*}{h^*}\frac{\partial w^*_{\mathrm{ev}}}{\partial r^*} - \frac{w^*_{\mathrm{ev}} t^*}{h^{*2}}\frac{\partial h^*}{\partial r^*}\right] ; \tag{33}$$

$$\frac{\partial\varphi}{\partial t^*} = (1 - \varphi_0)\left[\frac{w^*_{\mathrm{ev}}}{h^*} - \frac{w_{\mathrm{ev}} t^*}{h^{*2}}\frac{\partial h^*}{\partial t^*}\right] . \tag{34}$$

The rheological properties of polymer compositions depend on the concentration of solvents they contain and their solvent capabilities [17, 18]. The form of these dependences can be found experimentally [19].

Having specified the initial condition

$$h^*(r^*, 0) = h_0^*(r^*), \tag{35}$$

to determine the current film thickness along the disk radius we obtain a Cauchy problem for an equation in first order partial derivatives (24), which can be solved numerically by the finite difference method.

Thus, the relationships obtained above permit calculation of the basic hydrodynamic characteristics of liquid film flow over the surface of a rotating disk for a wide class of media, the rheological behavior of which is described by Shul'man's model. Possible mass-exchange phenomena and change in rheological properties and density of the medium have been considered.

The results of the study can be used to construct mathematical models of various technological processes and for determination of optimal construction elements and operating regimes for centrifugal apparatus under concrete technological conditions.

NOTATION

τ, shear stress; τ_0, limiting shear stress; K, medium consistency parameter; m, n, exponents; $z^* = z/H$, $r^* = r/R$, dimensionless coordinates along z- and r-axes respectively; $\tau^* = \tau/(\eta\omega)$, dimensionless shear stress; η, characteristic viscosity; ω, angular velocity of disk rotation; $\mathrm{Re} = \mathrm{P}\omega\mathrm{RH}/\eta$, Reynolds number; $\mathrm{H} \sim 10^{-3}$ (m), characteristic film thickness; $\mathrm{R} \sim 10^{-1}$ (m), characteristic radius; $\rho^* = \rho/\mathrm{P}$, P is the dimensionless and characteristic density of the medium; $w^* = \mathrm{H/R}\ w/\omega\mathrm{R}$; $u^* = u/(\omega\mathrm{R})$, dimensionless velocities in axial and radial directions; u, radial velocity component; w, axial velocity component; $\tau_0^* = \tau_0/(\eta\omega)$, limiting dimensionless shear stress; $h^* = h/H$, dimensionless film thickness; $t^* = t\omega$, dimensionless time; $K^* = K/(\eta^{\frac{m}{n}}\omega^{\frac{m}{n}-1})$, dimensionless consistency coefficient; w_{ev}, axial velocity component related to evaporation; j, diffusion flux density; ρ_s, solvent density; D, diffusion coefficient; c_0, saturated vapor concentration at surface of liquid film and temperature of surrounding medium; ν, kinematic viscosity of medium surrounding disk; $\mathrm{Pr_d} = \nu/\mathrm{D}$, diffusion Prandtl number; φ, relative solvent concentration in polymer composition; φ_0, relative solvent concentration at initial moment; ρ_n, density of nonvolatile portion of composition; $w^*_{ev} = j/(\rho_s \omega H)$, dimensionless evaporation rate; h_0, film thickness at initial moment.

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